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# Supersymmetry, Ultraviolet Finiteness and Grand Unification<sup>1</sup>

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*Abstract.* A general criterion is given for the vanishing of the  $\beta$ -functions in  $N=1$  supersymmetric gauge theories.

## 1 Introduction and Conclusions

Supersymmetry is well known to enforce cancellations of ultraviolet (UV) divergences.  $N=4$  super-Yang-Mills (SYM) theory for instance is devoid of any UV divergence [1, 2, 3], and this holds as well for a class of  $N=2$  SYM models [4]. Our aim is to present similar results [5, 6, 7] for  $N=1$  SYM theories in 4-dimensional spacetime. We mean here, by UV finiteness, the vanishing of all the  $\beta$ -functions, i.e. the non-renormalization of the coupling constants. To the contrary of other approaches [8, 9] which tend to complete UV finiteness, we don't require the (nonphysical) anomalous dimensions to vanish, i.e. infinite (unobservable) field amplitude renormalizations may still be present.

The physical interpretation of our results is scale invariance – or better, asymptotic scale invariance since masses may be present – but we shall consider the massless case [6] for the sake of simplicity.

We shall give a general criterion, which involves one-loop quantities only, for the vanishing of the  $\beta$ -functions at all orders of perturbation theory. The criterion is based, first, on a relation between the anomaly of the axial  $R$ -current and the scale anomaly – expressed by the  $\beta$ -functions – which follows from the axial current

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and the energy-momentum tensor belonging to the supercurrent multiplet of Ferrara and Zumino [10], and from the nonrenormalization theorem for the axial anomaly. The second ingredient is the fact that the Yukawa couplings must necessarily be functions of the gauge coupling constant, functions solving the reduction equations of Oehme and Zimmermann [11]. The criterion is quite general, it is independent of the renormalization scheme used and it does not rely on the existence of a regularization preserving both gauge invariance and supersymmetry<sup>3</sup>. The procedure is based on general results of renormalization theory [12].

The only restriction is the assumption that the gauge group is a simple Lie group. But this lets it remain very interesting in the framework of Grand Unified Theories, where it may lead to predictions for the mass spectrum in particular. Applications of the criterion may be found in the second of refs. [5] for a SU(6) model and, more interestingly, in [13, 7], for a realistic SU(5) model compatible with the Minimal Supersymmetric Standard Model at low energies, which predicts a top mass of  $(185 \pm 5)$  GeV (see also [9]).

Our criterion can actually be considered as the rigorous version of a formal argument already given by the authors of Ref. [1] for the case of  $N = 4$  SYM.

Let us finally mention that we restrict ourselves here to unbroken symmetry. The case of supersymmetry being broken by soft mass terms is under study [14] in the framework of the Wess-Zumino gauge. A superspace approach with complete UV finiteness is proposed in [9].

## 2 Super Yang-Mill Theory

A generic  $N=1$  supersymmetric gauge theory, with a simple Lie group  $G$  as gauge group, is given at the classical level by

1. Supermultiplets  $V^a$  of gauge fields, each containing in particular a gauge vector field  $A_\mu^a$  and a gaugino Weyl spinor field  $\lambda_\alpha^a$ , in the adjoint representation of the gauge group, as well as matter supermultiplets  $S_i$ , each containing in particular a scalar field  $\phi_i$  and a Weyl spinor field  $\psi_{i\alpha}$ , in some unitary representation  $R$ . The gauge fixing is implemented through Lagrange multiplier chiral supermultiplets  $B^a$  and ghost (antighost) chiral supermultiplets  $C_+^a$  ( $\bar{C}_-^a$ ) containing in particular the ordinary ghosts (antighosts)  $c^a$  ( $\bar{c}^a$ ). External fields (the “antifields”)  $A_\mu^{*a}$ ,  $\lambda_\alpha^{*a}$ ,  $\phi^{*i}$ ,  $\psi_\alpha^{*i}$ , etc. have to be introduced in order to control the renormalization of the BRS transformations given below, which are not linear.

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<sup>3</sup>In a generic scheme the Yukawa  $\beta$ -functions are not necessarily linear combinations of the matter anomalous dimensions – except in the one-loop approximation – but this has no consequence on our result, which concerns the  $\beta$ -functions only.

## 2. The BRS transformations

$$\begin{aligned}
sA_\mu^a &= (D_\mu c)^a + \cdots := \partial_\mu c + f^{abc} A_\mu^b c^c + \cdots, & s\lambda_\alpha^a &= -f^{abc} c^b \lambda_\alpha^c + \cdots, \\
s\phi_i &= -R_{ai}{}^j c^a \phi_j + \cdots, & s\psi_{i\alpha} &= -R_{ai}{}^j c^a \psi_{j\alpha} + \cdots, \\
sc^a &= -\frac{1}{2} f^{abc} c^b c^c + \cdots, & \cdots &.
\end{aligned} \tag{1}$$

## 3. The BRS invariant action

$$\begin{aligned}
\Sigma = \int d^4x \left( -\frac{1}{4g^2} F^{a\mu\nu} F_{a\mu\nu} - \frac{i}{g^2} \lambda^{a\alpha} \sigma_{\alpha\dot{\alpha}}^\mu D_\mu \bar{\lambda}_a^{\dot{\alpha}} + \overline{D^\mu \phi_i} D_\mu \phi^i - i\psi^{i\alpha} \sigma_{\alpha\dot{\alpha}}^\mu D_\mu \bar{\psi}_i^{\dot{\alpha}} \right. \\
\left. + \lambda_{(ijk)} \psi^i \psi^j \phi^k + \text{conj} + \cdots \right), \tag{2}
\end{aligned}$$

where  $g$  is the gauge coupling constant and the symmetric invariant tensors  $\lambda_{ijk}$  are the Yukawa coupling constants. We take both the gauge and the matter fields massless.

**Remark:** Our notations are very sketchy, the details may be found in the original literature [15, 5]. First, we have written everything in terms of component fields instead of superfields for the sake of transparency. Second, we have omitted the contributions of a lot of component fields, namely the auxiliary fields, the nonphysical components of the gauge superfield, as well as the Lagrange multiplier, ghost and antighosts fields, and the external fields. All these omissions are signalized by dots in (1), (2) and in the following. Third, many numerical coefficients have been skipped and arbitrarily replaced by the number 1.

## 3 The Scale Anomaly

The theory being massless is scale invariant. But this is generally true only in the classical approximation. Radiative corrections cause a breaking of this invariance, i.e. there is a scale anomaly. It is well known that the scale anomaly manifests itself as a nonvanishing trace of the energy-momentum tensor  $T_{\mu\nu}$  – which may be assumed to be traceless in the tree (i.e. classical) approximation<sup>4</sup>:

$$T_\mu^\mu = \beta_g (F^{\mu\nu} F_{\mu\nu} + \cdots) + \sum_{ijk} (\beta_{ijk} \psi^i \psi^j \phi^k + \cdots) + \cdots + O(\hbar^2), \tag{3}$$

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<sup>4</sup>We expand in the powers of Planck's constant  $\hbar$ , i.e. in the number of loops in Feynman graphs.

where

$$\begin{aligned}\beta_g &= \hbar b_0 + O(\hbar^2) , & b_0 &:= l(R) - 3C_2(G) , \\ \beta_{ijk} &= \hbar b_{ijk} + O(\hbar^2) , & b_{ijk} &:= \sum_{\text{cycl}(ijk)} \lambda_{ijn} \left( \lambda^{npq} \lambda_{pqk} - 2\delta_k^n g^2 C_2(R) \right)\end{aligned}\quad (4)$$

are the  $\beta$ -functions corresponding to the renormalizations of the gauge coupling  $g$  and of the Yukawa couplings  $\lambda_{ijk}$ , respectively.  $l(R)$  is the Dynkin index of the representation  $R$ . We have not explicitly written the contributions of the anomalous dimensions of the various fields.

**Remark:** Let us recall that the  $\beta$ -functions determine the behaviour of the effective coupling constants as solutions of the differential equations

$$\mu \frac{d}{d\mu} \bar{g}(\mu) = \beta_g(\mu, \bar{g}, \bar{\lambda}) , \quad \mu \frac{d}{d\mu} \bar{\lambda}_{ijk}(\mu) = \beta_{ijk}(\mu, \bar{g}, \bar{\lambda}) , \quad (5)$$

where  $\mu$  is the energy scale.

## 4 $R$ -Invariance and Axial Anomaly

A common feature of massless  $N=1$  supersymmetric theories is their invariance under the  $U(1)$  chiral transformation  $R$ :

$$A'_\mu = A_\mu , \quad \lambda'_\alpha = e^{-i\theta} \lambda_\alpha , \quad \phi' = e^{-i\frac{2}{3}\theta} \phi , \quad \psi'_\alpha = e^{i\frac{1}{3}\theta} \psi_\alpha , \quad \dots \quad (6)$$

The associated axial Noether current  $J_R^\mu(x)$

$$J_R^\mu = \bar{\lambda} \gamma^\mu \gamma^5 \lambda + \dots \quad (\text{Dirac notation}) \quad (7)$$

is conserved in the classical limit, but not in the quantum case due to the axial anomaly:

$$\partial_\mu J_R^\mu = r (\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + \dots) \quad (8)$$

The anomaly coefficient  $r$  has the remarkable property of being equal to the one-loop value of the gauge  $\beta$ -function (see (4)):

$$r = \hbar b_0 . \quad (9)$$

Moreover, due to a nonrenormalization theorem [16, 17]<sup>5</sup>, the one-loop value given here is exact, without any higher order contributions.

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<sup>5</sup>see [5] for the generalization to the present case of supersymmetric gauge theories.

## 5 The Supercurrent

We shall now see that the equality (9) between the  $R$ -anomaly coefficient and the one-loop value of the gauge  $\beta$ -function is not an accident. It is actually linked to the existence of a supermultiplet of gauge invariant operators: the supercurrent [10]  $\mathcal{J}$ :

$$\mathcal{J} := \left\{ J_R^\mu, \quad Q_\alpha^\mu, \quad T^\mu{}_\nu \right\}, \quad \dots, \quad (10)$$

made of the (classically) conserved currents associated to  $R$  invariance, supersymmetry and translation invariance, respectively. Note that the latter two remain conserved to all orders, to the contrary of the first one. Note also that the  $R$ -current  $J_R^\mu$  in (10) differs from the  $R$ -Noether current (7). But both coincide in the tree approximation:  $J_R^\mu = J_R^\mu + O(\hbar)$ . The point is that, starting from the unique classical Noether current  $J_{R(\text{class})}^\mu$ ,  $J_R^\mu$  is defined as the quantum extension which allows for the validity of a nonrenormalization theorem [17], whereas  $J_R^\mu$  is defined to belong together with the energy-momentum tensor to one supermultiplet – the supercurrent (10). These two requirements cannot be fulfilled simultaneously by a single current operator.

More interesting, there is a second supermultiplet (a chiral multiplet) containing, among others, the anomalies of the  $R$ -current  $J_R^\mu$  as well as the trace anomalies of the supersymmetry current and of the energy-momentum tensor. One can indeed prove the set of equations – which constitute the “supertrace identity”:

$$T^\mu{}_\mu = \text{Re } S, \quad \partial_\mu J_R^\mu = \text{Im } S, \quad \sigma_{\alpha\dot{\beta}}^\mu \bar{Q}_\mu^{\dot{\beta}} = S_\alpha, \quad (11)$$

where the anomalies in the right-hand sides belong to the chiral supermultiplet

$$\mathcal{S} = \{ \text{Re } S, \quad \text{Im } S, \quad S_\alpha, \quad \dots \} := \left\{ \beta_g F^{\mu\nu} F_{\mu\nu} + \dots, \quad \beta_g \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + \dots, \quad \beta_g \lambda^\beta \sigma_{\alpha\beta}^{\mu\nu} F_{\mu\nu} + \dots, \quad \dots \right\} \quad (12)$$

called the supertrace anomaly.

## 6 The Relation Between the Scale and the Axial Anomalies

Despite of the discrepancy between the axial current  $J_R^\mu$  obeying the anomalous conservation law (8) with a nonrenormalized coefficient  $r$  and the axial current  $J_R^\mu$  belonging to the supercurrent multiplet (10), whose anomaly coefficient is the  $\beta$ -function characterizing the scale anomaly, there is a relation between them [5]:

$$r = \beta_g(1 + x_g) + \beta_{ijk} x^{ijk} - \gamma_A r^A. \quad (13)$$

This equation relates  $r$  and  $\beta_g$  together with the Yukawa  $\beta$ -functions and some other coefficients:

the  $r^A$ 's, which are the *nonrenormalized* coefficients of the anomalies of the Noether currents associated to the chiral invariances of the superpotential,

the  $\gamma_A$ 's, which are some linear combinations of the anomalous dimensions of the matter fields,

$x_g$  and the  $x^{ijk}$ 's, which are radiative correction quantities we don't need to specify.

Let us emphasize that all the coefficients appearing in (13) are of order  $\hbar$  at least. Moreover,  $r$  and the  $r^A$ 's are strictly proportional to  $\hbar$ , i.e. strictly one-loop quantities due to the nonrenormalization theorems for the axial anomalies. An important remark is that the structure of this identity is *independent from the renormalization scheme*, although the individual coefficients – except the one-loop values of the  $\beta$ -functions – may be scheme dependent.

## 7 Ultraviolet Finiteness

It is clear from the second of eqs. (4) that the vanishing of the  $\beta$ -functions implies already at the one-loop approximation that the Yukawa coupling constants  $\lambda_{ijk}$  must be functions of the gauge coupling constant  $g$ . A necessary and sufficient condition for the existence of a similar relation to all orders is that the Yukawa coupling constants be formal power series  $\lambda_{ijk}(g)$ , in  $g$ , of the reduction equations[11]

$$\beta_g \frac{d\lambda_{ijk}}{dg} = \beta_{ijk} . \quad (14)$$

The identity (13) then allows us to give a general criterion for the ultraviolet finiteness in the sense of vanishing  $\beta$ -functions, i.e. of physical scale invariance:

**Theorem.** Consider an  $N=1$  super-Yang-Mills theory with simple gauge group. If

- (i) there is no gauge anomaly,
- (ii) the gauge  $\beta$ -function vanishes at one loop (see (4)):

$$b_0 := l(R) - 3C_2(G) = 0 , \quad (15)$$

- (iii) there exist solutions of the form

$$\lambda_{ijk} = \rho_{ijk} g , \quad \rho_{ijk} \in \mathbb{C} , \quad (16)$$

to the equations

$$\lambda^{jpq}\lambda_{pqk} - 2\delta_k^j g^2 C_2(R) = 0 , \quad (17)$$

(iv) These solutions are isolated and non-degenerate when considered as solutions of vanishing one-loop Yukawa  $\beta$ -functions (see (4)):

$$b_{ijk} = 0 , \quad (18)$$

then each of the solutions (16) can be uniquely extended to a formal power series in  $g$ , and the associated super-YM models depend on the single coupling constant  $g$  with a  $\beta$ -function which vanishes at all orders.

**Remarks:**

- (a) Hyp. (ii) is equivalent to the vanishing of the  $R$ -current anomaly (see (9)).
- (b) The expressions in (17) (Hyp. (iii)) are the anomalous dimensions of the matter fields in the one-loop approximation. Their vanishing implies the vanishing of the one-loop Yukawa  $\beta$  functions due to the second of eqs. (4). In fact Hyp. (iii) also implies the vanishing of the chiral anomaly coefficients  $r^A$  appearing in (13). The latter property is moreover a necessary condition for having  $\beta$ -functions vanishing to all orders.
- (c) Hyp. (iv) is a condition which guaranties the existence of a formal power series solution to the reduction equations (14).
- (d) It is shown in [18] that the hypotheses (i) to (iii) assure the vanishing of the  $\beta$ -functions in the two-loop approximation. Thanks to Hyp. (iv) we are able to extend the result to all orders.

**Proof of the theorem:** Inserting  $\beta_{ijk}$  as given by the r.h.s. of the reduction equations (14) into the identity (13) and taking into account the vanishing of the chiral anomalies  $r$  and  $r^A$ , we get for  $\beta_g$  an homogenous equation of the form

$$0 = \beta_g (1 + O(\hbar)) . \quad (19)$$

Its solution in the sense of the formal power series in  $\hbar$  is  $\beta_g = 0$ . Hence  $\beta_{ijk} = 0$  as well, due to (14).  $\square$

## Discussion

*Slavnov:* Does a solution to the reduction equations always correspond to a symmetry giving the same relation between the coupling constants? This seems to me necessary for the stability of the solution.

*O.P.:* This may happen in the case where the field content is compatible with a higher symmetry, like  $N=4$  supersymmetry [5]. But I hardly see such a symmetry at work in the  $SU(5)$  and  $SU(6)$  models mentioned in the Introduction.

*Kazakov:* These solutions are infrared stable but ultraviolet unstable.

*Kazakov:* Do you rely on some invariant regularization?

*O.P.:* No (see the Introduction).

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## References

- [1] M.F. Sohnius and P.C. West, *Phys. Lett.* B100 (1981) 245;
- [2] S. Mandelstam, *Nucl. Phys.* B213 (1983) 149;  
L. Brink, O. Lindgren and B. Nilsson, *Nucl. Phys.* B212 (1983) 401;
- [3] P.L. White, *Class. Quantum Grav.* 9 (1992) 413;
- [4] P.S. Howe, K.S. Stelle and P.C. West, *Phys. Lett.* B124 (1983) 55;  
P.S. Howe, K.S. Stelle and P.K. Townsend, *Nucl. Phys.* B236 (1984) 125;  
P.C. West, “Supersymmetry and Finiteness”, *Shelter Island II* 1983:127;
- [5] O. Piguet and K. Sibold, *Int. J. Mod. Phys.* A1 (1986) 913;  
C. Lucchesi, O. Piguet and K. Sibold, *Helv. Phys. Acta* 61 (1988) 321;
- [6] O. Piguet and K. Sibold, *Helv. Phys. Acta* 63 (1990) 71;
- [7] C. Lucchesi and G. Zoupanos, “All-order Finiteness in  $N=1$  SYM Theories: Criteria and Applications”, *Proceedings of the Workshop on “Recent Developments in theoretical Physics”*, CERN, Geneva, Dec. 1995, hep-ph/9604216;
- [8] A.V. Ermushev, D.I. Kazakov and O.V. Tarasov, *Nucl. Phys.* B281 (1987) 72;  
D.I. Kazakov, *Mod. Phys. Lett.* A9 (1987) 663;  
D.R.T. Jones, *Nucl. Phys.* B279 (1986) 153;
- [9] D.I. Kazakov, M.Yu. Kalmykov, I.N. Kondrashuk and A.V. Gladyshev, JINR E2-95-482, Dubna 1995, hep-ph/9511419;  
D.I. Kazakov, in these Proceedings;
- [10] S. Ferrara and B. Zumino, *Nucl. Phys.* B87 (1975) 207;

- [11] R. Oehme and W. Zimmermann, *Commun. Math. Phys.* 97 (1985) 569;  
R. Oehme, K. Sibold and W. Zimmermann, *Phys. Lett.* B147 (1984) 117;  
B153 (1985) 142;  
W. Zimmermann, *Commun. Math. Phys.* 97 (1985) 211;  
R. Oehme, *Prog. Theor. Phys. Suppl.* 86 (1986) 215; *Proc. of XVIIIth. Int. Workshop on High Energy Physics and Field Theory, Moscow-Protvino*, June 1995, hep-th/9511006;
- [12] O. Piguet and S.P. Sorella, “Algebraic Renormalization”, *Lecture Notes in Physics, Vol. m28 (Springer Verlag, Berlin, Heidelberg, 1995)*;
- [13] J. Kubo, M. Mondragon, M. Olechowski and G. Zoupanos, MPI-PH-95-133, Dec. 1995, hep-ph/9512435;
- [14] N. Maggiore, O. Piguet and S. Wolf, UGVA-DPT-1966-04-917, hep-th/9604002;  
C. Lucchesi, N. Maggiore, O. Piguet and S. Wolf, in preparation;
- [15] O. Piguet and K. Sibold, “Renormalized Supersymmetry”, series “*Progress in Physics*”, vol. 12 (*Birkhäuser Boston Inc., 1986*);
- [16] W.A. Bardeen, in *Proc. 16th. Conf. High Energy Physics*, Batavia 1972, Vol. 2, p.195;  
W.A. Bardeen, *Nucl. Phys.* B75 (1974) 246;  
S.-Y. Pi and S.-S. Shei, *Phys. Rev.* D11 (1975) 2946;
- [17] C. Lucchesi, O. Piguet and K. Sibold, *Int. J. Mod. Phys.* A2 (1987) 385;
- [18] A.J. Parkes and P.C. West, *Phys. Lett.* B138 (1984) 99;  
*Nucl. Phys.* B256 (1985) 340;  
D.R.T. Jones and L. Mezincescu, *Phys. Lett.* B138 (1984) 293;  
D.R.T. Jones and A.J. Parkes, *Phys. Lett.* B160 (1985) 267;  
A.J. Parkes, *Phys. Lett.* B156 (1985) 73;